Formation and Maintenance of Thermoclines in Temperate Lakes

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Temperate lakes generally exhibit a vertical thermal structure that is characterized by a nearly homothermal upper layer with a sharply defined lower boundary (the thermocline). At the start of the warming season, the fairly smooth temperature profile resulting from surface heating is transformed to one that displays a thermocline, and, at later times, the thermocline slowly descends into the deeper, colder layers of the lake. The mechanisms leading to the formation and maintenance of this structure are investigated analytically. It is shown, through an analytical model, that both the formation and the maintenance of the thermocline can be explained in terms of the nonlinear interaction between wind-induced turbulence and stable buoyancy gradients due to surface heating. Some preliminary qualitative observations are also given on the effects of powerplant thermal discharges on the thermal structure of stratified bodies of water.

Introduction

THE term "thermocline" was first proposed by Birge in 1897 (to describe the layer of intense temperature gradient which separates the almost homogeneous upper layer from the colder bottom waters of deep, stratified lakes). A thermal structure in which a well-mixed, warm upper layer is separated from a relatively colder bottom region is indeed found to be characteristic of all deep bodies of water in temperature latitudes, including lakes, rivers, and the ocean. A number of theoretical models have been proposed to explain this characteristic thermal structure, but a satisfactory explanation of the fundamental reasons for the formation and maintenance of the thermocline is still lacking. In the authors' view, the basic phenomena that are responsible for the formation and the maintenance of the thermocline are the addition or loss of heat at the surface, turbulent transport of heat from the surface to greater depths, and the striking interaction of the turbulence with the temperature gradients.

The present study aims to show that a satisfactory theory for the mechanism of formation and maintenance of thermoclines in a temperate lake has to take into account two essential aspects. Firstly, the mechanism of not only the formation but also the maintenance of a thermocline is an unsteady process, even when conditions above the body of water under consideration are steady. Therefore, a satisfactory theory for the thermocline must include this basic aspect. Secondly, the formation of a thermocline is by the nonlinear interaction between the wind-generated turbulence and the stable buoyancy gradients in the body of water under consideration. This nonlinearity, although it makes the equations difficult to analyze, is an essential feature of the interaction between the turbulence and prevailing temperature structure and, as such, has to be retained.

In recent years, there has been a renewal of interest in the thermal structure of lakes because of the "thermal pollution" problem. This problem is caused by the discharge of waste heat from electric generating plants into bodies of water and the subsequent degradation of the quality of these waters.[1–3]

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It should be pointed out that a quantitative understanding of the mechanisms of formation and maintenance of the thermal and current structures in stratified lakes is a necessary prerequisite to making proper assessments of the perturbing effects of powerplant discharges.

In the next section, the various important features of the observed thermal structure of temperate lakes is described briefly. Then a theoretical model, which accounts for many of the observed characteristics, is proposed, and the validity of the various assumptions and approximations used in the model is discussed in detail. The proposed model is employed to study key features of the stratification process by using a direct-integration scheme. Finally, some qualitative observations on the effects of thermal discharges are given.

Thermal Structure of Temperate Lakes

During early spring, most temperate lakes exhibit a nearly homothermal temperature distribution with a temperature of about 4°C (which is the temperature of maximum density for water) extending all the way to the bottom. As the weather above the lake begins to warm, the lake receives heat, mainly by solar radiation, at an increasingly rapid rate. During the early part of the warming season, the lake continues to remain nearly homothermal, since the heat that is received at the surface layers by solar radiation is transported to the deeper layers by wind-induced currents and turbulence. As the rate of heating of the lake continues to increase, the rate at which heat is received at the surface layers soon exceeds the rate of heat removal to the deeper layers, and the temperature of surface layers begins to increase. During this early period, the temperature decreases monotonically with increasing depth, with the bottom temperature remaining close to that at the end of vernal circulation.

Figure 1 shows some typical plots of the vertical distributions of temperature in Cayuga Lake, N. Y. during various parts of the stratification cycle. The plots given in Fig. 1 have been obtained from the monthly averages of the temperatures measured by Henson et al.[4] The initial isothermal distribution and the later monotonically decreasing distribution can be seen in (a) and (b) of Fig. 1.

As the heating continues, a point of inflection develops in the temperature profile, and a well-mixed upper layer, with relatively intense temperature gradients at its bottom boundary, is formed [as shown in (c) of Fig. 1]. The plane of the maximum temperature gradient is, of course, the thermocline. During the remainder of the heating period, the thermocline slowly descends into the deeper, colder layers of the lake. It should be noted that, once a thermocline forms, the deeper regions of the lake are relatively uninfluenced by changes in surface conditions.

As the lake attains its maximum heat content and subsequently begins to cool, the thermocline moves down rapidly into the deeper layers of the lake as wind mixing is now augmented by convective mixing because of surface cooling. The thermocline continues to move down rapidly as the well-mixed upper layers cool further, as shown in (d) and (e) of Fig. 1, until the whole lake again attains homothermy.

In the brief description of the observed features of stratification just given, two important points should be noted. The first is the transformation of a smooth, monotonically decreasing temperature distribution at the beginning of the warming season into a temperature distribution that exhibits a well-defined thermocline. The second is the continuous downward erosion of the thermocline into the deeper, colder layers of the lake. In the authors' view, a satisfactory theory for the development of thermoclines in temperate lakes must account for these important aspects.

Basic Relations and Boundary Conditions

In the previous section, it was pointed out that, although the vertical temperature profile at the start of the stratification season in a temperate lake is fairly smooth, a sharp interface (the thermocline) develops at later times. It is argued here that the thermocline is formed by the nonlinear interaction between the wind-induced turbulence in a lake and the buoyancy gradients produced by surface heating, and that this nonlinearity is an essential feature that has to be retained in a satisfactory theoretical model. It is also argued that the mechanism of not only the formation of the thermocline but also its maintenance is an unsteady process, and that the continuous downward erosion of the thermocline into the deeper layers of the lake is a necessary part of its sustenance.

In a large, relatively deep lake, the transport of any property takes place much faster in the horizontal directions than in vertical ones, so that one can assume that horizontal homogeneity exists in planes parallel to the surface. It should be emphasized that the preceding assumption can be valid only in an approximate sense, since, as pointed out by Wedderburn and Watson[5] a number of years ago, the existence of a current in the well-mixed upper layer, or the epilimnion, must necessarily involve an upward tilting of the isotherm toward the windward end of the lake. However, in the localized theoretical model being proposed here, the small horizontal nonhomogeneity due to the tilting of the isotherms is of no direct consequence, since it does not influence the mechanism of formation or maintenance of the thermocline.

Under the assumption of horizontal homogeneity, the equation describing the vertical transport of heat is

$$\partial T/\partial t = (\partial/\partial Z)[K_H(\partial T/\partial Z)] \quad (1)$$

where T is the temperature, t is the time, Z is the distance measured downward from the surface, and K_H is the eddy diffusivity for the vertical transport of heat. Molecular thermal diffusivity is not explicitly accounted for in Eq. (1), since it is in general smaller than the eddy diffusivity, and, if necessary, it can be incorporated into the definition of K_H. It should be noted that Eq. (1) is nonlinear, since, in general, the eddy diffusivity is a function of the thermal as well as current structure in the lake.

Boundary and Initial Conditions

Before discussing the dependence of the eddy diffusivity on the thermal and current structures, it is appropriate to consider the initial and boundary conditions that have to be used in conjunction with Eq. (1). In all of the calculations presented in the present paper, the initial condition will be taken as that corresponding to the end of spring homothermy; that

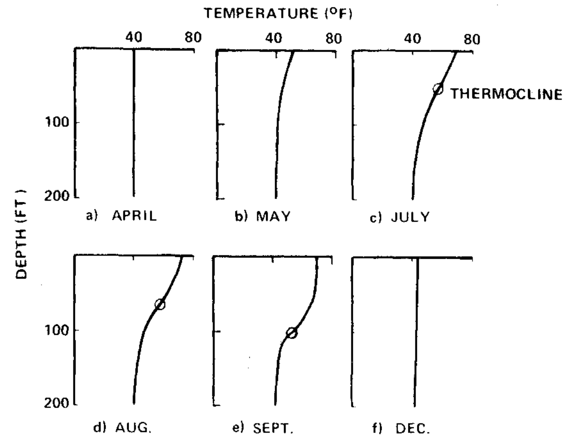

Fig. 1 Temperature structure of Cayuga Lake, 1952 (after Henson et al.[4]).

is, the initial condition will be taken as

$$T(Z, 0) = T_0 \quad (2)$$

where T_0 is the temperature of the lake at maximum spring homothermy. The boundary condition at the surface of the lake must describe the heat exchange between the lake and the atmosphere, and this can be written in the form[6]

$$q_s = -\rho c_p[K_H(\partial T/\partial Z)]_{z=0} = K(T_E - T_S) \quad (3)$$

where q_S is the heat flux (taken positive when downward), K is a heat-exchange coefficient, T_S is the surface temperature, and T_E is a fictitious surface temperature, called the equilibrium temperature, at which there would be no net heat transfer to or from the lake surface. The equilibrium temperature and the heat-transfer coefficient are both functions of the environmental conditions above the lake and can be expressed as functions of the wind speed, air temperature and humidity, and net incoming (sky and solar) radiation. Methods of evaluating T_E and K are described fully by Edinger and Geyer,[6] Sundaram et al.,[3] and Moore et al.[7]

Equation (3) is merely a statement of Newton's law of cooling. It should be pointed out that implicit in Eqs. (1) and (3) is the assumption that the bulk of the incoming solar radiation is absorbed within a small layer near the surface. This assumption is, in general, valid in most deep, turbid lakes. For example, as Ruttner[8] points out, the characteristic depth for absorption of the solar radiation in Seneca Lake, N. Y. is considerably smaller than the depth of the well-mixed upper layer.

Theories of the thermocline based on the so-called "internal-radiation-absorption model" have been given by Ertel[9] and by Dake and Harleman.[10] These theories account for turbulent diffusion only in an indirect and empirical way, and they are based on the assumption that the primary mechanism responsible for the formation of a thermocline is the differential absorption, at various depths, of the incoming solar radiation. If necessary, the feature of the differential absorption of the incoming radiation can be easily incorporated into Eqs. (1) and (2). However, it is felt that the inclusion of this additional feature will not add significantly to the conclusions derived in the present paper.

Forms of the Eddy Diffusivity

It was pointed out earlier that Eq. (1) is nonlinear, since the eddy diffusivity K_H is a function of the thermal as well as the current structure in the lake. One of the primary objectives of the present paper is to demonstrate that the nonlinearity of Eq. (1) is an essential feature of the interaction between wind-induced turbulence and buoyancy gradients due to surface heating and that, as such, it must be retained if a satisfactory theory for the thermal structure of a stratified

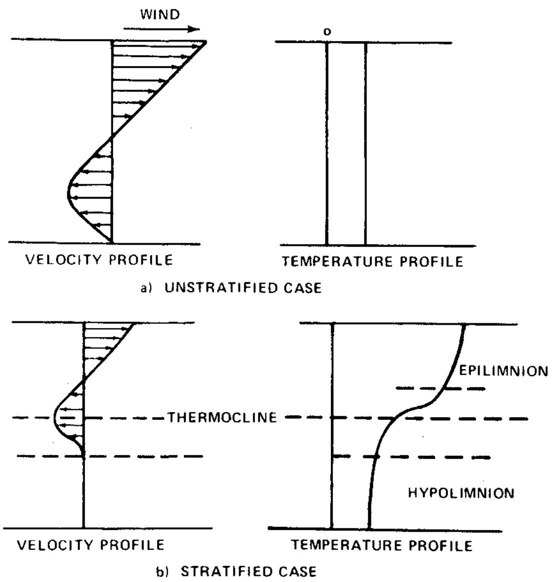

Fig. 2 Wind-induced currents in a lake.

lake is to be developed. Some authors[11] have used a constant value for the eddy diffusivity K_H in Eq. (1) when solving for the thermal structure of lakes. The practice of assuming a constant value for the eddy diffusivity is especially prevalent in oceanic-thermocline theories,[12,13] where phenomena such as Coriolis and advection effects, which are omitted in Eq. (1), are also included. In general, these theories predict a temperature distribution that decreases more or less uniformly with depth, and, in particular, they do not predict a thermocline separating a well-mixed upper layer from the cooler bottom layer.

Equation (1) can be used to predict a thermocline only if the interaction between the turbulence and buoyancy gradients is included. The interaction may be accounted for either explicitly or implicitly, as in two-layer models[3] in which the upper and lower parts of the lake are described by different, but constant, eddy diffusivities. It should be emphasized that the interaction between the turbulence and thermal structures is crucial in determining the structure of each. It is this interaction that makes the problem of predicting the effects of heated discharges on the thermal structure of stratified lakes difficult, since the heated discharges influence the thermal structure not only directly but also indirectly through their effect on the turbulence structure.[3]

The major mechanism by which turbulence is generated in the upper layers of a lake is the wind shear acting on the surface of the lake. Correspondingly, the buoyancy gradients are produced in the lake by the heat exchange, at the surface of the lake, between the environment and the lake. When the mean buoyancy field in the upper layers of a lake is statically stable, it tends to suppress the generation of wind-induced turbulence. Conversely, when the mean buoyancy field is statically unstable, it adds to the generation of wind-induced turbulence. The effects of the interaction between the turbulence and the buoyancy field on the structures of each other have been studied fairly extensively in connection with atmospheric and oceanic turbulence. Recent developments on the interaction between the turbulence and buoyancy fields in the lower atmosphere have been summarized by Lumley and Panofsky,[14] and those in the upper ocean have been summarized by Phillips[14] and Okubo.[16]

In general, the eddy diffusivity under arbitrary thermal stratification conditions can be written as the product of the eddy diffusivity under corresponding neutral stratification conditions and a function of an appropriate stability parameter characterizing the stratification. Thus, one can write

$$K_H = K_{H_0} f] \quad \text{(stability parameter)} \tag{4}$$

where K_{H_0} is the eddy diffusivity under identical environmental conditions but in the absence of stratification.

One of the more commonly used forms of the stability parameter is the gradient Richardson number[17] R_i, which is defined as

$$R_i = -\alpha_v g[(\partial T/\partial Z)/(\partial u/\partial Z)^2] \tag{5}$$

where α_v is the coefficient of volumetric expansion of water, g is the acceleration due to gravity, and u is the horizontal component of the current velocity. The denominator of the right-hand side of Eq. (5) represents the rate of production of turbulence by Reynolds stresses, whereas the numerator represents the rate of production or suppression of turbulence by the mean buoyancy field. The Richardson number is positive for a stable stratification, negative for an unstable stratification, and its absolute value increases with increasing stratification.

It was mentioned earlier that Eq. (1) is, in general, coupled to the equation describing the velocity field. It can be seen now that the coupling occurs through Eqs. (4) and (5). The only existing theory of the thermocline which accounts for the interaction between turbulence and stratification, through a coupling between the velocity and temperature fields, is the theory of Munk and Anderson.[18] These authors have assumed that the eddy diffusivity for heat and momentum can be expressed in the form given in Eq. (4), and they have extended the classical Ekman-Spiral analysis to account for the interaction between the turbulence and buoyancy fields. However, since the theory of Munk and Anderson is based on the assumption of steady state, certain inconsistencies arise at large depths. As the authors themselves point out, "it appears that the distributions of current and temperature cannot both be stationary at the same time."[18] Some interesting observations on the *unsteady* changes in the current structure that occur during a storm and their influence on the shape of the thermocline after the storm have been given by Mortimer.[19]

As the wind blows over the surface of a lake, the turbulence in the upper layers of the lake is generated both by mean shear and by breaking of the waves. As Phillips[15] points out, although the momentum flux from the air to the waves is only a small fraction of the momentum flux transferred to the current, the energy flux to the waves is usually comparable to or greater than the energy flux to the current. Several Russian workers, such as Dobrolonskii and Kitaigorodoskii (see Ichiye[20]), have indeed characterized the turbulence in the upper layers in terms of the predominant amplitudes, wavelengths, and periods of the surface waves.

Thus it is reasonable to suppose that the mechanical generation of turbulence in the upper layers of a lake can be characterized by the surface conditions alone without an explicit consideration‡ of the current structure. In other words, the Richardson number characterizing the interaction between the mechanically generated turbulence and the thermal structure can be taken, instead of that defined in Eq. (5), as

$$R_i = -\alpha_v g Z^2[(\partial T/\partial Z)/w^{*2}] \tag{6}$$

where $w^* = (\tau_S/\rho)^{1/2}$ is the friction velocity, τ_S is the surface shear stress induced by the wind, and ρ is the density of water. Forms of the Richardson number similar to that just given have been used by Pritchard[21] in his analysis of the dispersion of contaminants in tidal estuaries and have also been used recently by Kato and Phillips.[22]

Before we proceed to consider the form of function f in Eq. (4), one other point should be noted here. One of the more recent developments in atmospheric turbulence has

‡ It will be seen later that an implicit accounting of the current structure is nevertheless necessary for a proper characterization of the turbulence structure over the entire depth of the lake.

been the formulation of a "similarity theory" by Monin and Obukhov (see Ref 3). These authors postulate that the influence of thermal stratification on the turbulence at any depth Z can be characterized by the parameter (Z/L^*) alone, where L^* is a length scale given by

$$L^* = w^{*3}/[k\alpha_v g[q/\rho c_p)]$$

In this equation, k is the von Kármán constant (≈ 0.4) and C_p is the specific heat. The physical significance of the Monin-Obukhov length scale L^* is that it represents the depth at which the rate of production of turbulence by the Reynolds stress and that by the buoyancy flux are approximately the same. The significance of the Monin-Obukhov length in the stratification cycle of lakes, as well as its application to predicting the effects of thermal discharges, has been discussed in detail by Sundaram et al.[3]

In the literature,[3,14,16] a number of different forms have been used for the function of f in Eq. (4). Most of these relations are obtained by making various empirical assumptions; for example, some of the relations are extensions of Prandtl's mixing-length theory[23] to include the effects of stratification. Of the many existing relations, two have been chosen for the purposes of the present study as being typical ones. These relations are

$$K_H = K_{H_0}(1 + \sigma_1 R_i)^{-1} \tag{7}$$

$$K_H = K_{H_0}(1 - \sigma_2 R_i) \tag{8}$$

where σ_1 and σ_2 are empirical constants.

The first of the preceding relations was deduced by Rossby and Montgomery,[24] and the second relation was originally proposed by Holzman.[25] Kent and Pritchard[26] have tested Eqs. (7) and (8) with their observations in a coastal plain estuary, and their results seem to indicate that Eq. (7) fits the experimental data better than Eq. (8). However, within the scope of the accuracy of the measurements and the other unknowns involved, it is not possible to establish whether or not Eq. (7) is a more suitable form than Eq. (8).

In the present study, both Eqs. (7) and (8) have been used, in conjunction with Eq. (6), to express the eddy diffusivity K_H in Eq. (1) in terms of the local temperature gradient.

Turbulence in the Deeper Layers of a Lake

Equation (6) describes the eddy diffusivity due to the turbulence generated by surface wind stress and will not be valid for the deeper layers of a lake, since the mechanisms by which turbulence is generated in the deep layers are considerably different from those by which turbulence is generated in the upper layers. It has already been pointed out that the regions below the thermocline are little influenced by changes in surface conditions. It was also mentioned earlier that, although an explicit coupling between Eq. (1) and the current structure was not retained in the present study, an implicit accounting has to be included.

When a lake is unstratified, the wind-induced current structure in it will be as shown in Fig. 2a, with the entire lake being in circulation. However, when the lake is stratified, the wind-induced drift is confined to the upper layers, with the maximum return current occurring near the thermocline, as shown in Fig. 2b. This characteristic current structure was first pointed out by Wedderburn[27] and has since then been verified by a number of others.[3,28,29] It immediately follows from the current structure that there is no mechanism by which the wind stress at the surface can directly create turbulence in the hypolimnion. This point needs to be emphasized, since, in spite of all evidence to the contrary, some authors have assumed high values for the eddy diffusivity in the hypolimnion. The only mechanisms by which turbulence can be created in the hypolimnion are indirect ones,[3] such as degradation of internal waves and currents produced by internal seiches,[30] and water withdrawal.[31] None of the preceding indirect mechanisms is explicitly accounted for either in the present study or in any existing theory of the thermocline.

McEwen[32] has proposed a very simple method of demonstrating that the eddy diffusivities in the hypolimnion are small and nearly invariant with depth, and he has used this method to evaluate the values of the eddy diffusivities in the hypolimnion of Lake Mendota. This method has also been used on data obtained on various lakes by other authors,[3,28] who find that the eddy diffusivity is constant over an extended region of the hypolimnion (except close to the bottom of the lake), and that the values are quite small, being only a few times larger than molecular diffusivity.

It is clear that Eqs. (7) and (8), which predict that the value of the eddy diffusivity will approach K_{H_0} at large depth, cannot be valid in the hypolimnion, because the value of K_{H_0} is typically two or three orders of magnitude greater than the molecular diffusivity. Moreover, since the dominant processes, such as seiches, by which turbulence is produced in the hypolimnion are not accounted for in the present study, the following procedure was adopted to describe the eddy diffusivity over the entire depth of the lake. Equation (1) was assumed to be valid over the entire lake with the eddy diffusivity being described by Eq. (7) or (8), so long as the minimum value of the eddy diffusivity was greater than some specified value $\bar{\nu}$. When the minimum value of the eddy diffusivity becomes equal to or less than $\bar{\nu}$, Eq. (7) or (8) is assumed to be valid down to the point of minimum diffusivity, and the eddy diffusivity is assumed to remain constant at the value $\bar{\nu}$ below this point. The foregoing procedure is necessary to account implicitly for the observed change in current structure with the onset of stratification.

Integration of the Basic Equation

As discussed in the preceding sections, the analysis of the thermocline can be divided for convenience into two phases, that of formation and that of erosion of the thermocline into the hypolimnion. Both of these phases are embodied in solutions of Eq. (1) with boundary conditions and an appropriately selected thermal diffusivity. Dimensional analysis of Eq. (1) for a rather general form of nonlinearity provides specific information about the depth at which the thermocline can be expected to form and about the time at which formation will occur. Results of direct numerical integration of Eq. (1) for a specific form of the nonlinearity and for various boundary conditions at the surface are then presented and discussed.

If, for simplicity, the coefficient for thermal expansion α_v and the frictional velocity w^* are each taken to be constant during the heating process, then a constant $\mu = g\alpha_v/w^{*2}$ can be defined, and the Richardson number can be written as

$$R_i = -\mu Z^2(\partial T/\partial Z) \tag{9}$$

It should be noted that the dimensions of the constant μ are simply reciprocal length times reciprocal temperature. With Eq. (9) used for the stability parameter, Eq. (4) becomes

$$K_H = K_{H_0} f[\mu Z^2(\partial T/\partial Z)] \tag{10}$$

and Eq. (1) can be written as

$$\frac{\partial T}{\partial t} = K_{H_0}\frac{\partial}{\partial Z}\left[f\left(\mu Z^2\frac{\partial T}{\partial Z}\right)\frac{\partial T}{\partial Z}\right] \tag{11}$$

Solution of Eq. (11) requires specification of boundary conditions at the surface of the lake. A quite general condition relating the surface temperature and the heat flux has been given in Eq. (3). However, to derive an understanding of the nonlinear and unsteady aspects of the interaction under simplest conditions, solutions to Eq. (11) for steady boundary conditions were obtained first. The formation process and the erosion of the thermocline into the hypolimnion were ob-

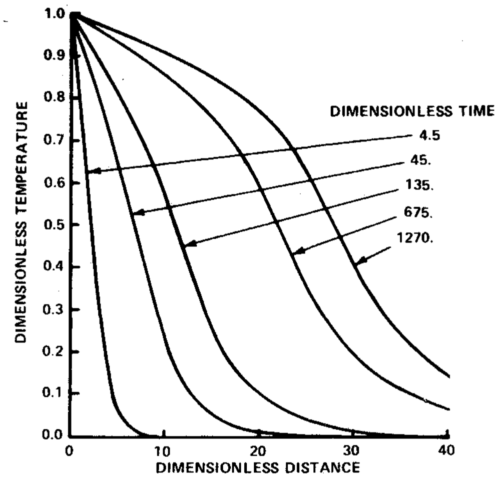

Fig. 3 Temperature calculated from Eq. (1) for a constant temperature at the surface using the thermal-diffusivity form proposed by Rossby and Montgomery.[24]

tained with a constant temperature T_1 imposed at the surface and with a constant heat flux q_0 imposed at the surface. With the understanding gained from these examples, interpretation of solutions with the more general boundary conditions will be simplified.

For the simple boundary conditions just discussed, an estimate of the time for formation and of the depth at which the thermocline forms can be determined by dimensional analysis of Eq. (11). The solution to Eq. (11) depends upon the only two parameters appearing in the equation, namely, K_{H_0} and μ. The dimensions of these two quantities are $[K_{H_0}] = [l^2/t]$ and $[\mu] = [1/lT]$, where $[l]$ is length, $[t]$ time, and $[T]$ temperature. Specification of a boundary condition, either constant temperature or a constant heat flux, then determines through dimensional analysis a characteristic length scale and a characteristic time scale. For example, specification of a constant temperature T_1 at the surface of the lake, for which the initial temperature is T_0, implies that the solution will depend only upon the two parameters of Eq. (11) and the temperature difference $T_1 - T_0$. From these three parameters, the length scale L and the time scale τ can be formed:

$$L = 1/\mu(T_1 - T_0) = w^{*2}/g\alpha_v(T_1 - T_0)$$
$$\tau = 1/K_{H_0}[\mu(T_1 - T_0)]^2 = w^{*4}/K_{H_0}g^2\alpha_v{}^2(T_1 - T_0)^2 \quad (12)$$

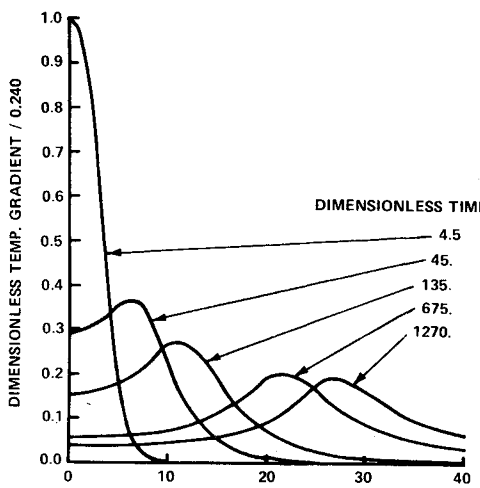

Fig. 4 Temperature gradient calculated for the conditions given in Fig. 3.

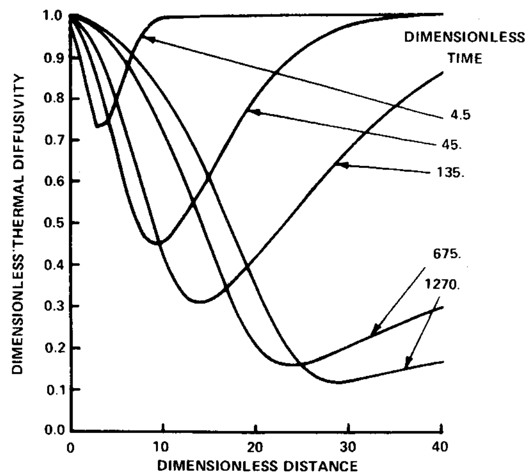

Fig. 5 Thermal diffusivity calculated for the conditions given in Fig. 3.

The length L represents the approximate depth at which nonlinear effects will become important, i.e., the depth near which a thermocline or a point of inflection in the temperature profile would be expected to form. Correspondingly, the time τ represents the approximate time at which a thermocline would be expected to form.

When a constant heat flux q_0 is imposed at the surface of the lake, the two parameters in Eq. (11) and $q_0/\rho c_p$ can be used to form a length scale and a time scale

$$L = (K_{H_0}\rho c_p/q_0\mu)^{1/2} = (K_{H_0}\rho c_p w^{*2}/q_0 g\alpha_v)^{1/2}$$
$$\tau = \rho c_p/q_0\mu = \rho c_p w^{*2}/q_0 g\alpha_v \quad (13)$$

The length scale L can be related simply to the Monin-Obukhov length scale L^* defined earlier. The length scale L, or the Monin-Obukhov length L^*, is a measure of the depth at which buoyancy effects and turbulence effects become comparable.

Equation (11) has been solved directly by numerical methods, with each of the two particular functional forms for K_H given in Eqs. (7) and (8). The numerical integration procedure used is a straightforward, formal generalization of the usual explicit finite-difference equation for the linear diffusion equation.[33] The spatial length over which the integration was performed was divided into equal increments, and the time step size was allowed to vary from step to step. Distance was measured in units of the spatial mesh size ΔZ and time in units of the time $(\Delta Z)^2/K_{H_0}$. For each calculation, ΔZ was chosen to be some fraction β of the dimensional length L, $\Delta Z/L = \beta$. The size of β, and therefore of ΔZ, was selected to obtain a desired resolution in the calculation.

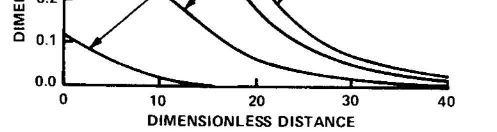

Fig. 6 Temperature calculated from Eq. (1) for a constant heat flux at the surface using the thermal-diffusivity form proposed by Rossby and Montgomery.[24]

Therefore, the unit of time is also a multiple of the dimensional time τ, $(\Delta Z)^2/K_{H0} = \beta^2\tau$. In addition, the deviation of the temperature from its initial value is measured in units of a reference temperature. For the condition of a constant temperature T_1 at the boundary, the reference temperature is $T_1 - T_0$; for a constant heat flux applied at the surface, the reference temperature is $(q_0/\rho c_p K_{H0}\mu)^{1/2}$.

Many numerical integrations of Eqs. (11) have been performed under a variety of conditions. Results of only three of the numerical calculations will be shown. The first two cases were computed using form (7) for the thermal diffusivity, one case in which a constant temperature boundary condition was imposed at the surface and the other case in which a constant heat flux was imposed at the surface. The third case was calculated with form (8) chosen for the thermal diffusivity and with a constant temperature imposed at the boundary. The figures were prepared directly from computer-generated plots. The constants by which the ordinates are reduced in Figs. 4, 6, and 7 were provided by the computer.

Figures 3–5 show the results of a numerical integration of Eq. (11) with thermal diffusivity form (7) when the temperature at the surface is maintained at a constant value. Figure 3 displays temperature as a function of distance or depth below the surface of the lake, whereas Fig. 4 displays the temperature gradient and Fig. 5 the thermal diffusivity. The solution for temperature of the linear form of Eq. (11), with $K_H = K_{H_0}$, can be written as a complementary error function of the argument $Z^2/K_{H_0}t$. For this case, the temperature gradient is a monotonically decreasing function of distance for any fixed time. The plots corresponding to the first time, 4.55 units, on each figure are closely approximated by the solution to the linear equation.

By the second time shown on Figs. 3–5, 45.0 units, the nonlinear behavior can be distinguished. The temperature plot displays an inflection point, which shows up on the temperature-gradient profile as a maximum. This point identified the thermocline. The length scale ΔZ and the time scale $(\Delta Z)^2/K_{H_0}$ are determined to be $\Delta Z = 0.2L$ and $(\Delta Z)^2/K_{H_0} = 0.04\tau$, where L and τ are given by Eq. (12).

At the third time on Figs. 3–5, 135 units, the temperature profile displays a fully developed thermocline or inflection point. The temperature gradient shown on Fig. 4 has a distinct maximum, which occurs at a position deeper than that found at the preceding time. The minimum in the thermal diffusivity of Fig. 5 is smaller and occurs at a larger distance than that found at the preceding time.

The last two plots on Figs. 3–5 675 and 1270 units, show the slow, continual progression of the thermocline to larger depths. The temperature profiles show that the epilimnion increases in depth even with the constant temperature imposed at the lake surface. The temperature gradient clearly displays the downward progression of the thermocline as time increases. The position at which the minimum in the thermal diffusivity occurs, the position where the heat transfer downward is most inhibited, also propagates to greater depths, and the minimum value of the diffusivity continues to decrease with increasing time.

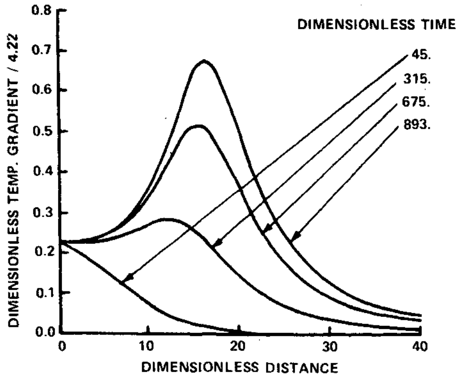

Fig. 7 **Temperature gradient calculated for the conditions given in Fig. 6.**

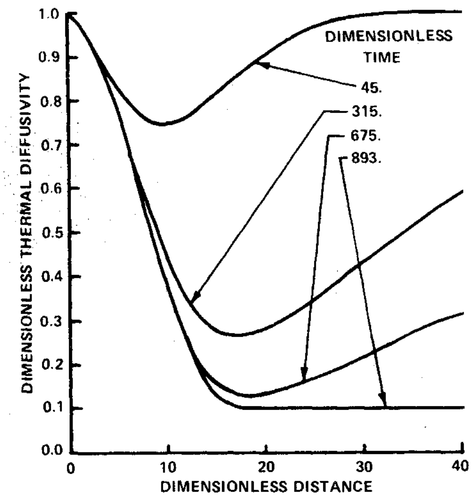

Fig. 8 **Thermal diffusivity calculated for the conditions given in Fig. 6.**

Figures 6–8 show the results of numerical solution of Eq. (11) with thermal diffusivity form (7) with a constant heat flux imposed at the surface. Figure 6 shows temperature as a function of distance for four times during the heating, whereas Fig. 7 shows the temperature gradient and Fig. 8 the thermal diffusivity. The plots corresponding to the first time, 45 units, on each figure are closely approximated by the solution to the linear equation. Both the temperature and the temperature gradient are monotonically decreasing functions of position, whereas the thermal diffusivity deviates by little more than 20% from constant.

By the second time on Figs. 6–8, 315 units, the nonlinear effects have begun to manifest themselves. The temperature profile displays an inflection point, and the temperature gradient displays a maximum. The minimum in the thermal diffusivity is much smaller and occurs deeper than that found at the first time. For this calculation, the length is expressed in units of $\Delta Z = 0.1L$ and the time in units of $(\Delta Z)^2/K_{H_0} = 0.01\ \tau$, where L and τ are given by Eq. (13).

At the third time on Figs. 6–8, 675 units, the temperature profile displays a more distinct inflection point, and the maximum in the temperature gradient is much larger. The position at which the maximum in the temperature gradient occurs has moved to a slightly larger depth. The reduction in the value of the thermal diffusivity below the surface impedes the downward transport of heat, and consequently the temperature at the surface is found to increase more rapidly than for the solution of the linear equation. At the final time, 893 units, the cutoff of the thermal diffusivity has taken place, and the temperature rise at the surface is even more rapid; the maximum of the temperature gradient has continued to increase. The nature of the cutoff of the diffusivity is clearly seen on Fig. 8.

Figure 9 shows the results of numerical integration of Eq. (11) with thermal diffusivity form (8) when the temperature at the surface is maintained at a constant value. Temperature as a function of distance is shown for four times during the computation. The length scale ΔZ and the time scale $(\Delta Z)^2/K_{H_0}$ for this calculation are $\Delta Z = \frac{1}{15}L$ and time $(\Delta Z)^2/K_{H_0} = \frac{1}{225}\ \tau$, where L and τ are given in Eq. (12). The plot corresponding to the first time, 45 units, displays the form characteristic of the solution of the linear equation. By the second time, 135 units, the temperature profile displays an

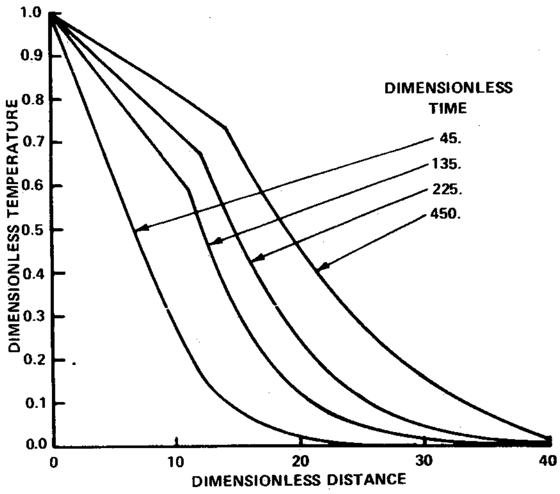

Fig. 9 Temperature calculated from Eq. (1) for a constant temperature at the surface using the thermal-diffusivity form proposed by Holzman.[25]

abrupt change near 11 units of depth. This change in temperature gradient is characteristic of all calculations performed with the thermal diffusivity of form given in Eq. (8). For this form, the cutoff discussed earlier occurs abruptly, yielding discontinuities in both the thermal diffusivity and in the temperature gradient. However, the temperature and the heat flux remain continuous in the calculations. The abrupt change in the temperature gradient signals the formation of the thermocline due to nonlinear effects. The subsequent erosion of the thermocline to greater depths is shown clearly in the last two curves at times 225 and 450.

Comparison of Fig. 9 with Fig. 3 shows that the qualitative behavior obtained from the second form of the diffusivity is the same as that obtained from form (7). Both the formation process, which occurs near a depth determined by length L and about a time determined by τ, and the erosion of the thermocline into the hypolimnion are found to occur by the unsteady and nonlinear processes in qualitatively the same fashion, even in the presence of the steady imposed temperature at the boundary.

Effect of Thermal Discharges

It was pointed out earlier that powerplant discharges affect the thermal structure of a lake directly, as well as through their influence on the turbulence structure of the lake. Thus, the thermal discharges affect the formation of the thermocline. It can be seen from Eqs. (12) and (13) that, if the effect of the thermal discharges is to cause an increase either in temperature or in heat flux, then the time of formation of the thermocline, as well as its depth, are reduced. Moreover, it can be seen that, for a given increase either in the temperature or the heat flux, the effect on the time of formation is more than the effect on the depth.

When the solution of the model equation over an entire cycle becomes available, the effect of thermal discharges on the complete cycle can be computed by including the addition of heat flux due to the discharge into Eq. (3). It may be noted that the effect of thermal discharges over the entire cycle may be viewed as causing an increase in the equilibrium temperature T_E. The results for an entire cycle, including the effects of thermal discharges, will be given in a later paper.

Concluding Remarks

The major objective of the present paper was to demonstrate that the formation and maintenance of the thermocline in a stratified lake can be explained in terms of the interaction between wind-induced turbulence and buoyancy gradients due to surface heating. It was demonstrated that, although the vertical temperature profile at the start of the stratification season in a temperate lake is fairly smooth, a sharp interface (the thermocline) develops because of the nonlinear interaction between the turbulence and the temperature structure. It was also shown that the continuous downward erosion of the thermocline into the deeper layers of the lake is a necessary part of its sustenance. The preceding conclusions were confirmed by considering the simple cases of a half-space, initially at a uniform temperature, which is subjected to a sudden increase in either surface temperature or heat flux. In the first case, the temperature distributions were initially similar to the error-function distribution for the linear case, but nonlinear effects soon came into play, and a thermocline was formed some distance below the surface. The thermocline also propagated steadily away from the surface.

In the case in which a constant heat flux was applied, the initial solution again corresponded to a linear solution, with a thermocline forming at later times. The numerical solution for this case seemed to indicate that the thermocline slowly progressed downward with increasing time, but it was not possible to establish whether or not an asymptotic state was reached.

The calculations were carried out for two different forms of the eddy diffusivity, namely, the form proposed by Rossby and Montgomery[24] and the form proposed by Holzman.[25] The qualitative nature of the conclusions was found to be the same for both cases, although in the latter case the thermocline appeared to be much sharper than in the former case. When the entire stratification cycle is considered, other effects not included in the present study are also important. For example, the present study did not include the variations in the conditions over the lake or the variation in the volumetric coefficient for expansion with temperature. Therefore, no quantitative comparisons of the present theory with observations were made. However, the qualitative behavior of the solutions obtained here is similar to those observed by Turner and Kraus[34] and by Kato and Phillips.[22]

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